M 100 36

THE OPEN UNIVERSITY

Mathematics Foundation Course Unit 36

Mathematical Structures

The Open University

Mathematics Foundation Course Unit 36

MATHEMATICAL STRUCTURES

Prepared by the Mathematics Foundation Course Team

Correspondence Text 36

The Open University Press

Open University courses provide a method of study for independent learners through an integrated teaching system including textual material, radio and television programmes and short residential courses. This text is one of a series that make up the correspondence element of the Mathematics Foundation Course.

The Open University's courses represent a new system of university level education. Much of the teaching material is still in a developmental stage. Courses and course materials are, therefore, kept continually under revision. It is intended to issue regular up-dating notes as and when the need arises, and new editions will be brought out when necessary.

Further information on Open University courses may be obtained from The Admissions Office, The Open University, P.O. Box 48, Bletchley, Buckinghamshire.

The Open University Press
Walton Hall, Bletchley, Bucks

First Published 1971

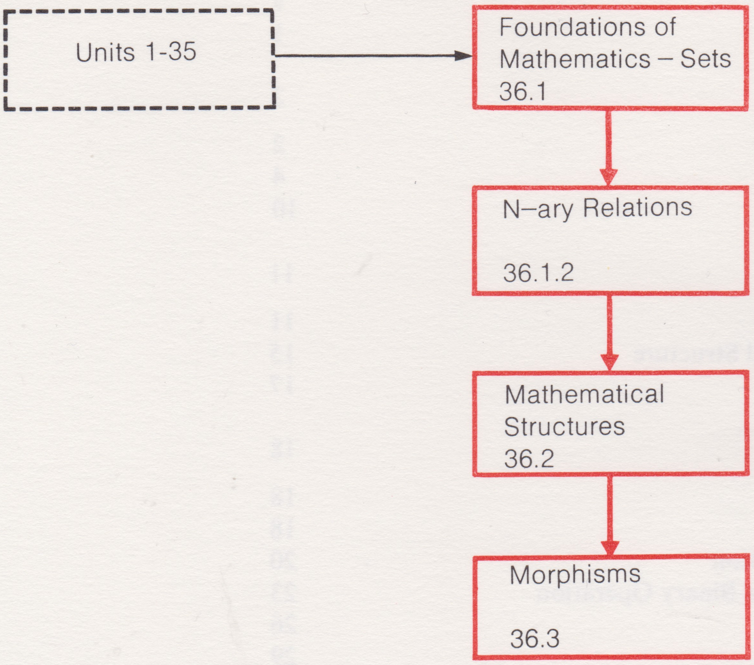
Printed in Great Britain by
J W Arrowsmith Ltd, Bristol 3

SBN 335 01035 0

Contents

		Page
	Structural Diagram	iv
	Glossary	v
	Notation	v
	Bibliography	v
	Introduction	1
36.1	**Foundations of Mathematics**	2
36.1.1	Sets	2
36.1.2	*n*-ary Relations	4
36.1.3	Summary	10
36.2	**Mathematical Structures**	11
36.2.1	Finding a Definition	11
36.2.2	Vector Space as a Mathematical Structure	15
36.2.3	Summary	17
36.3	**Morphisms**	18
36.3.0	Introduction	18
36.3.1	Structures with a Relation	18
36.3.2	Structures with a Binary Operation	20
36.3.3	Structures with a Relation and a Binary Operation	23
36.3.4	Morphisms of a Vector Space	26
36.3.5	The Real Numbers	29
36.3.6	Morphisms of a Mathematical Structure	32
36.4	**An End or a Beginning?**	33

Structural Diagram



Glossary

Page

Terms which are defined in this glossary are printed in CAPITALS.

EXTERNAL RELATION	An n-ARY RELATION $$\rho \subseteq S_1 \times S_2 \times \cdots \times S_n$$ is EXTERNAL if it is not INTERNAL.	7
INTERNAL RELATION	An n-ARY RELATION $$\rho \subseteq S_1 \times S_2 \times \cdots \times S_n$$ is INTERNAL if the sets $S_1, S_2, \cdots, S_n$ are the same.	7
MATHEMATICAL STRUCTURE	A MATHEMATICAL STRUCTURE is a set, together with any INTERNAL n-ARY RELATIONS defined on it.	12
MORPHISM	See the detailed discussion in the text.	19, 21, 32
n-ARY RELATION	An n-ARY RELATION is a subset, ρ, of the Cartesian product $S_1 \times S_2 \times \cdots \times S_n$.	4

Notation

The symbols are presented in the order in which they appear in the text.

$\mathscr{P}(S)$	The set of all subsets of a set S.	3
$\varnothing$	The empty set.	3
ρ	A general n-ary relation.	4
$S_1 \times S_2 \times \cdots \times S_n$	The Cartesian product of the sets $S_1, S_2, \ldots, S_n$.	4
$\wedge$	The symbol of conjunction (read as "and").	8
$(S; \rho_1, \rho_2, \ldots, \rho_m)$	The mathematical structure consisting of the set S together with m internal n-ary relations, $\rho_1, \rho_2, \ldots, \rho_m$, defined on S.	12
$\oplus_2$	The operation "add, then take the remainder on division by 2".	13
$\vee$	The symbol of alternation (read as "or").	14
$\mathscr{V}$	The union of the set of vectors of a vector space and the set of real numbers.	16
$\mathscr{F}$	The set of differentiable functions with domain R.	21
A/ρ	The quotient set of A by the relation ρ.	23
$[x]$	The equivalence class to which x belongs.	23
n	The natural mapping.	23
$\bar{\rho}$	The natural equivalence relation.	24

Bibliography

P. R. Halmos, *Naive Set Theory* (Van Nostrand, 1960). For those who would like to follow a rigorous development of the foundations of mathematics, this book (Chapters 1–12) provides one of the most readable accounts of the topic. For the devoted mathematician, Halmos' book is compulsory reading.

Mighty is the charm
Of those abstractions to a mind beset
With images and haunted by himself,
And specially delighted unto me
Was that clear synthesis built up aloft
So gracefully; even then when it appeared
Not more than a mere plaything, or a toy
To sense embodied: not the thing it is
In verity, an independent world,
Created out of pure intelligence.

William Wordsworth
The Prelude, Bk. 6.

36.0 INTRODUCTION

36.0

Introduction
⋆ ⋆

This is the last unit of the Mathematics Foundation Course, and it will not introduce any essentially new material — we shall simply be pouring the same pint into a different pint-pot. However, before you read this correspondence text, we strongly advise you to watch the television programme associated with this unit.

We shall look back over the previous units and draw together some of the threads of the mathematics that they contain. You have met many ideas in the Foundation Course, and we now want to organize those ideas so that you have an overall perspective of the study of mathematics in general and of the Foundation Course in particular. However, this unit is not intended to be a review of the course, and it does not contain any exhaustive survey of the topics covered. What we hope it does, is show that we can express quite a number of mathematical ideas in terms of a few elementary concepts. This is not just a neat exercise; it can lead to a heightened understanding of the various mathematical objects we manipulate. The fact that so many apparently diverse concepts can be expressed in terms of so few basic ideas, allows us to establish relationships which not only help us to grasp the concepts at the initial stages, but also often help us to tackle problems in a new field by methods which are familiar to us from past experience. A typical example of this is the basic idea of the *kernel of a morphism*, which we have used to solve equations in various situations.

We can summarize what we shall do in this unit in the following way. We shall attempt to describe mathematics (or rather, the objects with which mathematics is concerned) in a rather special way. Our description has three characteristics.

(i) It is systematic in the sense that it treats many different aspects of mathematics in the same way.
(ii) It is an attempt to analyse mathematics in terms of the simplest possible concept — that of a *set*, which thus becomes the foundation (for the purpose of this development) of mathematics.
(iii) A number of further concepts such as *relation* and *operation* are developed in terms of sets. Once defined, these concepts become entities in their own right, which can be used for further development.

The combination of a set with a number of relations forms what is known as a *mathematical structure*, which is, as it were, the bare framework of a branch of mathematics. Given the structure, we can deduce its properties.

36.1 FOUNDATIONS OF MATHEMATICS

36.1

36.1.1 Sets

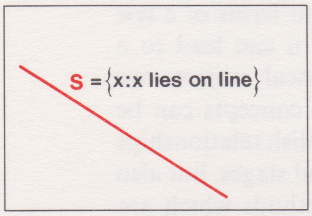
36.1.1

Main Text

A basic idea in mathematics is that of *set*, so it seems sensible to start our investigation of mathematical structure by contemplating sets. We have an intuitive notion of what a set is, and in *Unit 1, Functions*, we introduced the necessary mathematical language and notation so that we had a formal way of writing down these intuitive ideas. We write

$$S = \{x : x \text{ has the property } P\}.$$

For example, if x represents any point and P is the property of lying on a particular line, l, then we think of S as the set of points which lie on the line l.



The set S is a subset of a set T if each element of S is an element of T i.e. $S \subseteq T$.

If $S \subseteq T$ and $T \subseteq S$, then we say S and T are *equal* and we write $S = T$.

To define a set, we would need a more fundamental notion that was generally accepted. We could, in theory, (and there are some people who do, in practice) go on searching for more primitive concepts from which to construct our mathematical dictionary and shorthand; but we would have to stop somewhere and accept some concept without further investigation if our primary object was to make progress in the other direction.

Let us say, then, that we choose to make our primitive concept that of set. We pointed out in *Unit 17, Logic II* that our intuitive notion of set is inadequate in some discussions. There we met Russell's paradox concerning "the set of all sets". We can, however, get round the difficulties presented by Russell's paradox quite satisfactorily, if we are prepared to restrict our discussion of sets to those which are subsets of some set U, the universe of discourse. In practice, most mathematicians feel that *set* forms an adequate primitive concept, and they leave the rarefied atmosphere of more primitive notions to those who study the foundations of mathematics as a subject in its own right. In what follows we shall try to build up the necessary language in order to describe, in terms of sets alone, some of the concepts that you have met in the Foundation Course. By going through this building-up process, we hope that you will get some feeling for the unity of mathematics, because we shall place special emphasis on the similarities between the concepts that are mentioned.

At this point, a word of caution is necessary. As we go along, putting more and more words into our mathematical dictionary, we shall be skating over some of the problems that ought to be faced in a fundamental treatment. For the most part, these problems will probably not be noticeable, and so we shall not bother you with them.

To illustrate the sort of point that we shall skate over, consider the idea of mathematical induction. Suppose $P(n)$ is a hypothesis which depends on the positive integer n, and we prove:

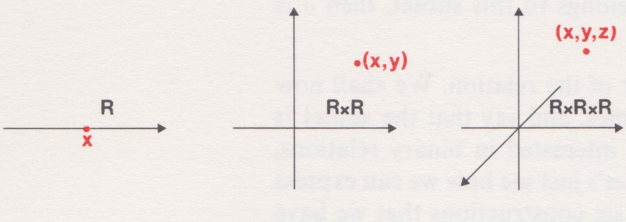
(i) $P(1)$ is true,
(ii) $P(n)$ is true $\Rightarrow P(n+1)$ is true, for any positive integer n.

It now seems obvious that $P(n)$ is true for all positive integers n. That is what intuition would say — and for the purposes of this unit, we shall let intuition be the judge. But we pointed out in *Unit 17, Logic II*, that we need to make this one of our formal mathematical assumptions. We called it the *Axiom of Mathematical Induction.*

So we start with the idea of *set.* Where can we go from here? There are two important constructions that we use to get more sets from any given set:

(i) The Cartesian product

This is the construction whereby we take a given set, consider, say, n copies of it, and then form n-tuples of elements of the set.



In fact, we do not restrict ourselves to making copies of the *same* set. In *Unit 3, Operations and Morphisms*, we defined the Cartesian product of two sets P and Q to be

$$P \times Q = \{(p, q) : p \in P \text{ and } q \in Q\}.$$

Here, the important property is that the pair (p, q) is *ordered*: the pairs (p, q) and (q, p) are not the same unless p and q are the same. If p and q are the same, we write $p = q$.

(ii) The set of all subsets of a given set

This is the construction whereby, given any set S, we form $\mathscr{P}(S)$, the set of all subsets of S. For example, if

$$S = \{a, b, c, d\},$$

then

$$\begin{aligned}\mathscr{P}(S) = \{&\varnothing,\\ &\{a\}, \{b\}, \{c\}, \{d\},\\ &\{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\},\\ &\{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\},\\ &\{a, b, c, d\}\}\end{aligned}$$

Notice that we can now regard any subset of S as an element of the set of all subsets, $\mathscr{P}(S)$. In symbolic form, this is written as follows.

If $T \subseteq S$ then

$$T \in \mathscr{P}(S).$$

36.1.2 *n*-ary Relations

36.1.2

Main Text
* *

Let us now see how we can define some of the terms that we met earlier in the course, using only the three notions:

set,
Cartesian product,
set of all subsets.

Probably the most general concept we met was that of a *relation*. Although *relation* was discussed formally only in *Unit 19*, *Relations*, in fact it encompasses many other ideas. We gave the following definition of a binary relation in *Unit 19*.

Given two sets A and B (which may be the same), any subset of $A \times B$ defines a relation from A to B. If (a, b) belongs to this subset, then a is *related* to b.

We called the subset itself the *solution set* of the relation. We shall now identify the subset and the relation it defines, and say that the subset *is* the relation. In *Unit 19*, we were mainly interested in binary relations, although we did define an n-ary relation. Let's just see how we can express the general n-ary relation in terms of the set constructions that we have discussed.

Definition 1
* * *

Given the sets $S_1, S_2, \ldots, S_n$, an n-ary relation on these sets, ρ, is defined to be a subset of the Cartesian product of $S_1, S_2, \ldots, S_n$. That is,

$$\rho \subseteq S_1 \times S_2 \times \cdots \times S_n.$$

If the sets $S_1, S_2, \ldots, S_n$ are all the same set S, then we say that ρ is defined on S.

In practice, we find that this definition is too general, and we need to make restrictions in the form of certain properties which we require the relations to have. For example, equivalence and ordering relations are binary relations which have certain properties. But we shall also see that binary operations, functions, etc. can be regarded as relations with certain properties.

Example 1

Example 1

A unary relation ρ on a set S is just a subset of S, that is, it is an element of the set of all subsets of S. ■

Example 2

Example 2

We can express quite a sophisticated idea, which we met for the first time in *Unit 35*, *Topology*, in terms of relations.

Let S be any set, and let ρ be a unary relation on $\mathscr{P}(S)$, that is,

$$\rho \subseteq \mathscr{P}(S).$$

ρ is a collection of subsets of S. We make the following restrictions on ρ:

(i) $\varnothing$ and $S \in \rho$;
(ii) any union* of elements of ρ is an element of ρ;
(iii) any intersection* of a finite number of elements of ρ is an element of ρ.

Under these restrictions, ρ is called a *topology* on S. So a topology on S is a special type of unary relation on $\mathscr{P}(S)$. ■

* Here, union and intersection can be thought of as binary operations on $\mathscr{P}(S)$. We shall see how to express union and intersection as n-ary relations in Exercise 36.2.1.2.

Example 3

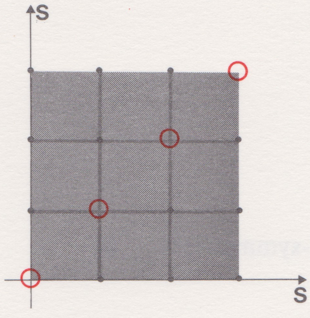
If $S_1 = S_2 = S$, then a binary relation ρ on S is a subset of $S \times S$. We have found the following restrictions on the elements of this subset useful:

(i) $\forall_x (x, x) \in \rho \qquad (x \in S)$.

If we write this restriction in the notation of *Unit 19, Relations*, we have

$$\forall_x x \rho x \qquad (x \in S).$$

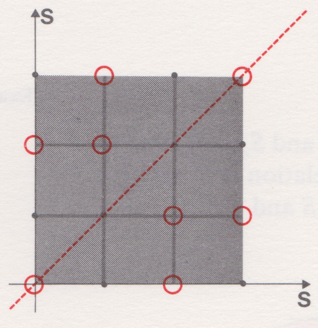
If a relation has this property, we say that it is reflexive. This property is depicted in the following example:



In the diagram, the elements of the Cartesian product marked by red rings are those belonging to the relation ρ. We use this convention in subsequent diagrams.

(ii) Whenever $(x, y) \in \rho$, then $(y, x) \in \rho$.

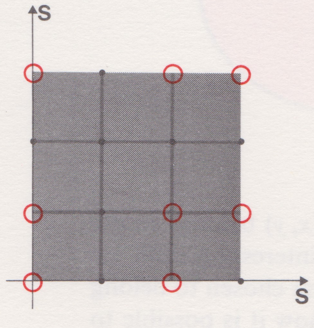
If a relation has this property, we say that it is symmetric. This property is depicted in the following example:



Notice the symmetry about the red line.

(iii) Whenever $(x, y) \in \rho$ and $(y, x) \in \rho$, then $x = y$.

If a relation has this property, we say that it is anti-symmetric. This property is depicted in the following example.



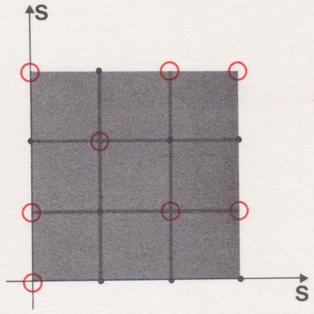
(iv) Whenever $(x, y) \in \rho$ and $(y, z) \in \rho$, then $(x, z) \in \rho$.

If a relation has this property, we say that it is transitive. This property is not easy to depict. ■

Exercise 1

Exercise 1
(3 minutes)

(i) Referring to Example 3(iii), why does the following diagram not depict an anti-symmetric relation on S?



(ii) Can a binary relation be both symmetric and anti-symmetric? ■

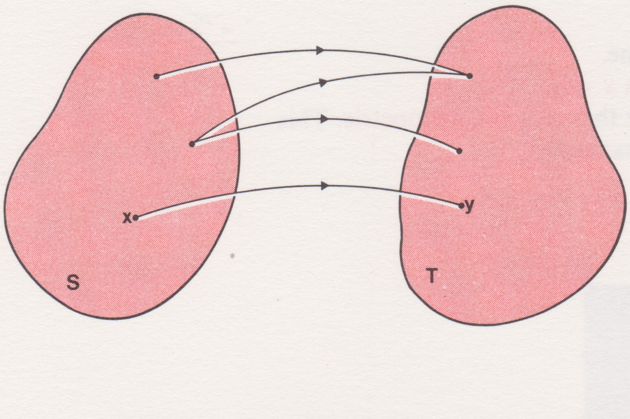
Discussion

The study of binary relations on a set S leads us to consider certain relations which combine some of the four properties mentioned in Example 3. The following types of relation were discussed in *Unit 19*:
equivalence relations, which have properties (i), (ii) and (iv);
partial order relations, which have properties (i), (iii) and (iv);
total order relations, which are partial order relations which also have the property that for all $x, y \in S, x \neq y$, either $(x, y) \in \rho$, or $(y, x) \in \rho$.

Example 4

Example 4

Here we consider a binary relation on the sets $S_1 = S$ and $S_2 = T$, that is, a subset of $S \times T$. The general effect of this type of relation is to set up a correspondence between the elements of the two sets S and T.



An element x is related to an element y if the pair (x, y) belongs to the relation ρ. But the relationship is not particularly interesting until we impose some restrictions on the way in which pairs are chosen to belong to ρ. In *Unit 3*, *Operations and Morphisms*, we saw how it is possible to think of a mapping as a subset of the Cartesian product of two sets. So we can now define mapping (and also function) using a special type of binary relation.

(i) The relation ρ defines a mapping $S \longrightarrow T$, if for each $x \in S$, there is at least one $y \in T$ for which

$$(x, y) \in \rho.$$

(ii) The relation ρ defines a function $S \longrightarrow T$, if for each $x \in S$, there is one and only one $y \in T$ for which

$$(x, y) \in \rho. \qquad \blacksquare$$

Discussion ⋆⋆

We have now introduced quite a few new words into our vocabulary and we have defined them only in terms of sets, subsets and Cartesian products. Each new word has been associated with some particular restriction placed on the way in which a subset can be chosen from a given set.

You may wonder why we have made a distinction between the binary relations of the type considered in Example 3 and those of the type considered in Example 4. Certainly the concepts defined are very different. The main reason for this difference lies in the fact that for Example 3 we restricted our relations to those of the type

$$\rho \subseteq S \times S,$$

i.e. the relation considered was a subset of the Cartesian product of a given set *with itself*, whereas, in Example 4, we imposed no such restriction. There we had relations of the type

$$\rho \subseteq S \times T,$$

i.e. the relation considered was a subset of the Cartesian product of two *possibly different sets*. (Notice that, in Example 4, we do not rule out the case $S = T$.)

We can think of relations of the type

$$\rho \subseteq S \times S$$

as being "internal," for the restrictions that we make relate elements of S with other elements of S, and this relationship is basically an internal one. In fact, it is this sort of relation which can often be used to tell us something about the structure of the set S.

On the other hand, we can think of relations of the type

$$\rho \subseteq S \times T \qquad (T \neq S)$$

as being "external", for here we are making restrictions on the way in which elements of S can be related to elements of a different set, T. Mappings and functions can correspond to internal or external binary relations.

Definition 2 ⋆⋆

Definition 3 ⋆⋆

In general, we shall refer to an n-ary relation as an internal relation, if all the n sets in the Cartesian product from which the relation is derived are the same. All other relations are external.

The next step is to consider ternary relations. These are the relations of the form

$$\rho \subseteq S \times T \times U.$$

Following our discussion of internal and external binary relations, we might go on to consider all the possible sorts of ternary relations, internal and external. But for the purposes of this unit we find that we have plenty of food for thought if we restrict our attention mainly to internal ternary relations, i.e. relations of the form

$$\rho \subseteq S \times S \times S.$$

The mathematical concept that we most closely associate with an internal ternary relation of the form

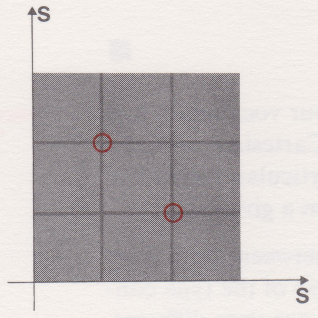
$$\rho \subseteq S \times S \times S$$

(*continued on page 8*)

Solution 1

Solution 1

(i) Consider the two points indicated on the following graph.



We see that we have two pairs for which

$$(x, y) \in \rho \quad \text{and} \quad (y, x) \in \rho$$

but $x \neq y$.
So this relation is not anti-symmetric.

(ii) Yes. For example, on the set Z, the relation

$$(x, y) \in \rho \quad \text{if } x = y$$

is both symmetric and anti-symmetric. ■

(*continued from page 7*)

is that of a closed binary operation. This is because the process that is represented by such an operation, $\circ$, involves three elements

$$x, y, x \circ y,$$

which all belong to the same set.

Thus the elements of ρ are the ordered triples $(x, y, z) \in S \times S \times S$, where

$$z = x \circ y.$$

Now there is an important restriction that we must place on ρ in order that it should represent a closed binary operation on S. It is:

for all $x, y \in S$, there is *one and only one* $z \in S$ for which $(x, y, z) \in \rho$.

Or, put another way,

for all $x, y \in S$, there is an $(x, y, z) \in \rho$,

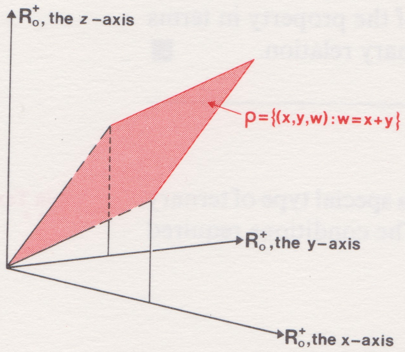
and

$$(x, y, z) \in \rho \wedge (x, y, z_1) \in \rho \Rightarrow z = z_1.$$

Example 5

Example 5

Let $S = R_0^+$ (where R_0^+ denotes the set of positive real numbers together with 0). If we consider the binary operation of addition (which is closed on S) as a ternary relation, then the corresponding ρ is a subset of $R_0^+ \times R_0^+ \times R_0^+$, as shown in the following diagram.

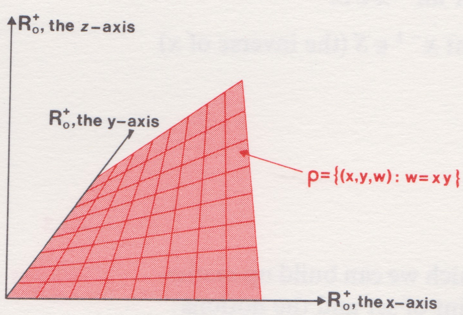


■

Example 6

Example 6

As in Example 5, suppose that $S = R_0^+$. Then the binary operation of multiplication is closed on S, and we can depict the corresponding ternary relation as follows.



■

Exercise 2

Exercise 2
(5 minutes)

Express the associative property for $\circ$ in terms of the ternary relation ρ.

(HINT: We want to express

$$x \circ (y \circ z) = (x \circ y) \circ z$$

for all $x, y, z \in S$, in terms of ρ. Do it bit by bit. For example, if $w = x \circ y$, then

$$(x, y, w) \in \rho.$$

Then if $w \circ z = a$,

$$(w, z, a) \in \rho.$$

Etc.)

■ (*continued on page 10*)

Solution 2

Solution 2

We can express the associative property by the following restriction.

If $(x, y, w) \in \rho$ and $(y, z, v) \in \rho$, then, for some element $a \in S$,

$$(x, v, a) \in \rho$$

and

$$(w, z, a) \in \rho.$$

This is so complicated, that obviously it is not useful as a practical test for associativity: for this purpose we prefer to think of the property in terms of a binary operation rather than in terms of a ternary relation. ■

(*continued from page 9*)

Main Text

We see that we can consider a group as a set S with a special type of ternary relation $\rho \subseteq S \times S \times S$. (See *Unit 30, Groups I.*) The conditions required of $\circ$ if $(S, \circ)$ is to be a group are as follows.

(i) $\circ$ is closed;
(ii) $\circ$ is associative;
(iii) there is an identity element $e \in G$ such that for all $a \in G$

$$a \circ e = a = e \circ a;$$

(iv) for any element $a \in G$, there is an inverse element $a^{-1} \in G$ such that

$$a \circ a^{-1} = e.$$

These restrictions can be rephrased to refer to the ternary relation, ρ, as follows:

(i) For all $x, y \in S$, there is one and only one $z \in S$ for which $(x, y, z) \in \rho$.
(ii) See Exercise 2.
(iii) There is an element $e \in S$ (the identity) such that

$$(e, x, x) \in \rho \quad \text{and} \quad (x, e, x) \in \rho \quad \text{for all} \quad x \in S.$$

(iv) For every element $x \in S$, there is an element $x^{-1} \in S$ (the inverse of x) such that

$$(x, x^{-1}, e) \in \rho.$$

36.1.3 Summary

36.1.3

Summary

In this section, we have looked at the way in which we can build up mathematical ideas from the very simple starting point of *set* and the notions:

Cartesian product $(S_1 \times S_2 \times \cdots \times S_n)$,

set of all subsets of S $(\mathscr{P}(S))$.

We defined an n-ary relation to be a subset of

$$S_1 \times S_2 \times \cdots \times S_n.$$

Two types of n-ary relation can be distinguished:

(i) an internal n-ary relation;
(ii) an external n-ary relation.

The type of ternary relation that we have been discussing is derived from the Cartesian product of a set S with itself, that is, an internal relation. It tells us something about the way in which the elements of one given set relate to each other, and, in that sense, it is descriptive of the structure of the set.

In the next section we shall see how we can use the ideas developed in this section to define a *mathematical structure*.

36.2 MATHEMATICAL STRUCTURES

36.2

36.2.1 Finding a Definition

36.2.1

Discussion
★ ★

In section 36.1, we met the ingredients we require to define a *mathematical structure*. In the television programme associated with this unit, we define a *mathematical structure* to be *a set, together with any relations and binary operations defined on it*. Let's examine the implications of this particular definition in terms of the previous arguments.

The first point to notice is the use of the term relation. We have been looking at many kinds of relation in the last section — for example, we saw how we could describe a closed binary operation as a particular kind of ternary relation. Clearly, the first problem to present itself is a linguistic one. What do we mean when we use the term *relation*?

The answer to this question comes in two parts.

(i) This part of the answer may begin to sound like an author's apology, but it is not intended to be so. Our original aim was to base our mathematical ideas upon the primitive concept of *set*. As we proceeded we introduced terms into our mathematical dictionary — Cartesian product, *n*-ary relation, mapping, function, equivalence relation, *etc*. Now it is very much in the spirit of mathematics to define new terms from old ones, and, once definitions have been made, to abandon the old in favour of the new. For example, we defined a mapping in the following way:

   the binary relation $\rho \subseteq S \times T$ defines a *mapping* $S \longrightarrow T$ if for each $x \in S$, there is at least one $y \in T$ for which $(x, y) \in \rho$.

   So rather than reproduce the latter mouthful each time we want to refer to a mapping, we simply substitute the term *mapping* in its place. This process leads to an economy of word and thought which is essential if a mathematical sentence or conversation is to be intelligible. The point here is that, once a definition has been made, we must be free to use the defined term whenever it is appropriate. But the full use of the newly defined term in no way subjugates the process which led to its definition to a trivial role. For in every statement containing a newly defined term, we rely totally on the plausibility of the process which gave rise to it.

(ii) In *Unit 19*, *Relations*, we used the term *relation* on S for the particular case of a binary relation ρ for which

$$\rho \in S \times S.$$

   That is, we referred to an internal binary relation simply by the term *relation* (on S). In practice, this is the accepted use of the word in most mathematical text books, and so, for the rest of this unit, we shall use the term in this sense. If we want to refer to a relation in the more general sense in which we have been using it so far, we shall use the term *n*-ary relation.

So much for *relation* in the definition of a mathematical structure: it is an internal binary relation.

The other term is binary operation. We have seen how we can think of a (closed) binary operation as a special kind of ternary relation

$$\rho \subseteq S \times S \times S,$$

that is, an internal ternary relation. If, therefore, we want to find a general definition of a mathematical structure, it would seem that we should do well to say that it is a set, together with any internal *n*-ary relations defined on it. And in practice this idea is perfectly adequate to cover most eventualities. We can now make the following definition.

A mathematical structure is a set together with any internal n-ary relations defined on it. **Definition 1** ★★★

Note that this general definition includes the definition given in the television programme.

We write

$$(S; \rho_1, \rho_2, \ldots, \rho_m)$$ **Notation 1** ★★

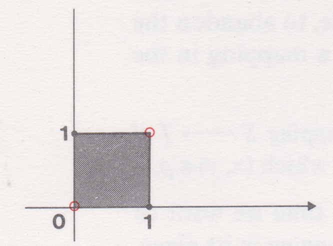
to denote the mathematical structure of the set S, together with m internal n-ary relations $\rho_1, \rho_2, \ldots, \rho_m$ defined on it.

Example 1 **Example 1**

Consider the set $\{0, 1\}$. Let us add a relation (i.e. an internal binary relation) to this set. There are many possible relations; we choose ρ_1 to be the simple equivalence relation of equality. That is,

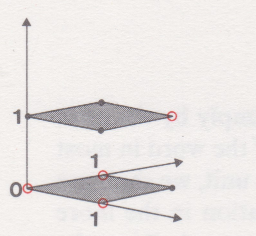
$$(x, y) \in \rho_1 \Leftrightarrow x = y.$$

Here, ρ_1 is the subset of $\{0, 1\} \times \{0, 1\}$ comprising the pairs (0, 0) and (1, 1).



The set ρ_1

Now let's add the binary operation of multiplication to the set. The ternary relation ρ_2 which represents this operation is depicted on the following diagram.



The set ρ_2

From the diagram, we see that the elements of ρ_2 are the four triples:

(0, 0, 0)

(0, 1, 0)

(1, 0, 0)

(1, 1, 1).

These have been chosen because each of them has the property that if $x, y \in \{0, 1\}$, then $(x, y, z) \in \rho_2$, where $z = x \times y \in \{0, 1\}$.

We now have a set, a binary operation defined on the set, and a relation defined on the set. In other words, we have a mathematical structure which we denote by

$$(\{0, 1\}; =, \times) \quad \text{or} \quad (\{0, 1\}; \rho_1, \rho_2). \qquad \blacksquare$$

Exercise 1

Exercise 1
(3 minutes)

Consider the set $\{0, 1\}$, together with the internal n-ary relations ρ_1 and ρ_2, where

(i) ρ_1 is the ternary relation corresponding to $\oplus_2$: the operation "add, then take the remainder on division by 2";
(ii) ρ_2 is the binary relation $\leqslant$ in its usual sense for real numbers.

Find the sets ρ_1 and ρ_2 and draw the graphs which depict them. ■

Example 2

Example 2

Consider the set of real numbers R. The following are internal binary relations which can be defined on R:

$$=, >, <, \geqslant, \leqslant.$$

Just a few of the internal ternary relations—i.e. binary operations—which we could define on R are:

$$+, -, \times, \circ, \square$$

where

$$\circ : (x, y) \longmapsto \exp(x + y) \qquad (x, y) \in R \times R,$$

$$\square : (x, y) \longmapsto x^2 + y^2 \qquad (x, y) \in R \times R.$$

The set R, together with the internal n-ary relations given above, forms a mathematical structure. We would denote it by:

$$(R; =, >, <, \geqslant, \leqslant, +, -, \times, \circ, \square).$$

No doubt it has some interesting properties! ■

Exercise 2

Exercise 2
(5 minutes)

(i) Show how the binary operation of "union" on the set $\mathscr{P}(S)$ of all subsets of S can be expressed as an internal ternary relation on $\mathscr{P}(S)$, with certain properties.
(ii) Show how the unary operation of "complement" on $\mathscr{P}(S)$ can be expressed as an internal binary relation, with certain properties. ■

Exercise 3

Exercise 3
(3 minutes)

Following the definition of an n-ary operation (*Unit 3, Operations and Morphisms*, section 3.1.5) show how an n-ary operation can be expressed as an $(n + 1)$-ary relation. ■

Solution 1

Solution 1

(i) We are looking for all triples (x, y, w), where $x, y, w \in \{0, 1\}$, which satisfy the property

$$x \oplus_2 y = w.$$

They are:

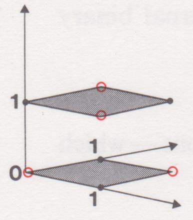
$$(0, 0, 0),$$

$$(1, 0, 1),$$

$$(0, 1, 1),$$

$$(1, 1, 0),$$

which we can depict as follows:



(ii) We are looking for pairs (x, y), where $x, y \in \{0, 1\}$, which satisfy the property

$$x \leqslant y.$$

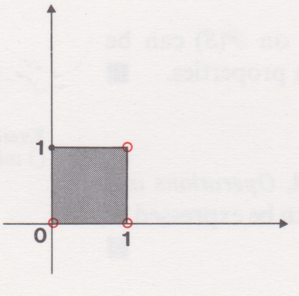
They are:

$$(0, 0),$$

$$(0, 1),$$

$$(1, 1),$$

which we can depict as follows:



■

Solution 2

Solution 2

(i) The elements of the subset ρ which represents $\cup$ have the form

$$(A, B, C),$$

where $A, B \in \mathscr{P}(S)$ and $C = A \cup B$. Thus the ternary relation which expresses this is

$$\rho \subseteq \mathscr{P}(S) \times \mathscr{P}(S) \times \mathscr{P}(S),$$

with the restriction

$$(A, B, C) \in \rho \quad \text{if and only if} \quad C = \{x : x \in A \lor x \in B\}.$$

(ii) This is much the same as part (i). But this time

$$\rho \subseteq \mathscr{P}(S) \times \mathscr{P}(S),$$

with the restriction

$$(A, B) \in \rho \quad \text{if and only if} \quad B = \{x : x \in S \wedge x \notin A\}. \quad \blacksquare$$

Solution 3

Solution 3

An n-ary operation is defined to be a function

$$f: \underbrace{A \times A \times \cdots \times A}_{n \text{ times}} \longrightarrow B.$$

This means that we can define the n-ary operation by specifying the $(n+1)$-ary relation

$$\rho \subseteq \underbrace{A \times A \times \cdots \times A}_{n \text{ times}} \times B,$$

with the property that for every n-tuple $(a_1, a_2, \ldots, a_n) \in A \times A \times \cdots \times A$, there exists a unique $b \in B$ such that

$$(a_1, a_2, \ldots, a_n, b) \in \rho.$$

If the operation is closed, we can take $B = A$ and the relation is internal.

$\blacksquare$

36.2.2 Vector Space as a Mathematical Structure

36.2.2

Main Text
★★

In this section we look at one type of mathematical structure in detail. In *Unit 22, Linear Algebra I*, we discussed vectors and the ways in which we combine them with one another and also with scalars. Towards the end of the unit, we put the ideas together under one heading — *vector space.* To define a vector space, we gave ten axioms:

1 $\underset{\sim}{v}_1 + \underset{\sim}{v}_2$ is a unique element of V (V is closed for addition)

2 $\underset{\sim}{v}_1 + (\underset{\sim}{v}_2 + \underset{\sim}{v}_3) = (\underset{\sim}{v}_1 + \underset{\sim}{v}_2) + \underset{\sim}{v}_3$ (addition is associative)

3 $\underset{\sim}{v}_1 + \underset{\sim}{v}_2 = \underset{\sim}{v}_2 + \underset{\sim}{v}_1$ (addition is commutative)

4 There is an element in V, which we call $\underset{\sim}{v}_0$, such that

$$\underset{\sim}{v} + \underset{\sim}{v}_0 = \underset{\sim}{v}$$

5 $\alpha\, \underset{\sim}{v}$ is an element of V

6 $\underset{\sim}{v} + (-1)\underset{\sim}{v} = \underset{\sim}{v}_0$

7 $\alpha(\underset{\sim}{v}_1 + \underset{\sim}{v}_2) = (\alpha\underset{\sim}{v}_1) + (\alpha\underset{\sim}{v}_2)$

8 $(\alpha + \beta)\underset{\sim}{v} = \alpha\underset{\sim}{v} + \beta\underset{\sim}{v}$

9 $(\alpha\beta)\underset{\sim}{v} = \alpha(\beta\underset{\sim}{v})$

10 $1 \times \underset{\sim}{v} = \underset{\sim}{v}$.

We can break down the set of axioms into three parts:

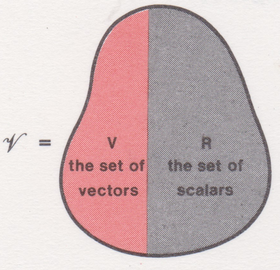
(i) axioms 1–4, 6 define $(V, +)$ to be a (commutative) group;
(ii) axiom 5 defines the way in which scalar multiplication operates;
(iii) axioms 6–10 are the restrictions on the way in which the operations of addition and multiplication by a scalar interact.

The important property to notice is that a vector space consists of not only a set of vectors, but also a set of scalars. In this course we have taken this set of scalars to be the real numbers. But we could take any set which has properties corresponding to the properties Re (1)–Re (4) of the real

numbers which we gave on page 1 of *Unit 6* — i.e. what mathematicians call a *field*. Again, we shall not go into the general situation, but take R as the set of scalars.

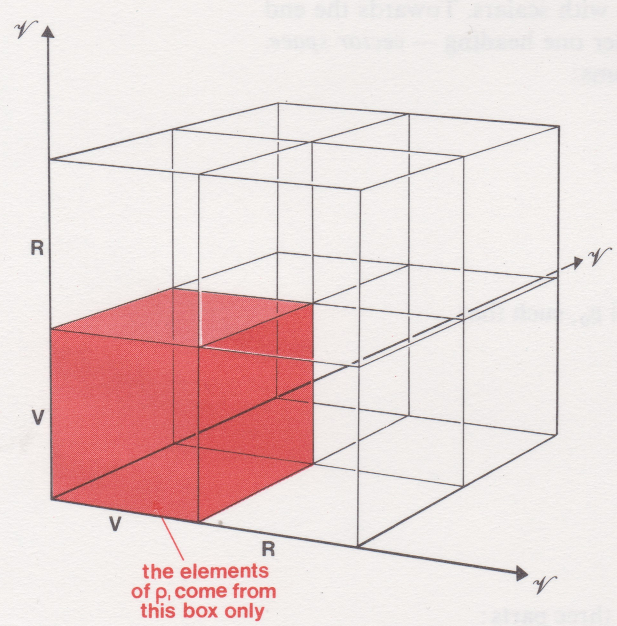
How then does a vector space fit into our general scheme of a mathematical structure? The interaction between vectors and scalars as expressed in axiom 5, does not lead to an internal ternary relation on V or on R. We can, of course, modify our definition of mathematical structure to allow relations other than internal ones, but this is not necessary. The answer is really quite simple. We consider just *one* set which is formed by taking the union of the set of vectors with the real numbers. We'll denote it by $\mathscr{V}$.

Notation 1
★ ★



We can now express addition of vectors and multiplication of a vector by a scalar in terms of two ternary relations, ρ_1 and ρ_2, on $\mathscr{V}$.

Looking more closely at ρ_1, which represents the addition of vectors, it is a subset of the Cartesian product $\mathscr{V} \times \mathscr{V} \times \mathscr{V}$, but each element of ρ_1 consists only of an ordered triple of vectors.

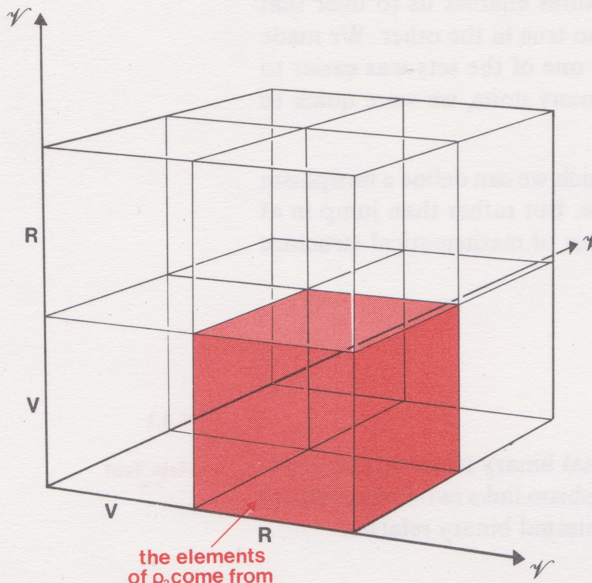


We have

$$(\underset{\sim}{v}_1, \underset{\sim}{v}_2, \underset{\sim}{v}_3) \in \rho_1 \quad \text{if and only if} \quad \underset{\sim}{v}_3 = \underset{\sim}{v}_1 + \underset{\sim}{v}_2$$

In *Unit 22*, we thought of addition of vectors as a binary operation on V, but now, with our extended set $\mathscr{V}$, we can no longer do this. The addition of vectors is *not* a binary operation on $\mathscr{V}$, because it is not defined for *every* pair of elements of $\mathscr{V}$. But this is no great loss, since we are able to

express addition of vectors in terms of an internal ternary relation. We can also express scalar multiplication in terms of a ternary relation, ρ_2. Each element of ρ_2 is an ordered triple consisting of a scalar, a vector, and a vector.



We have

$$(\lambda, \underset{\sim}{v}_1, \underset{\sim}{v}_2) \in \rho_2 \quad \text{if and only if} \quad \underset{\sim}{v}_2 = \lambda \underset{\sim}{v}_1$$

With this definition of scalar multiplication, we have made progress from the situation that we had in *Unit 22*. This is because, strictly speaking, we cannot think of scalar multiplication as a binary operation. However, it is perfectly respectable to think of it as a ternary relation on $\mathscr{V}$. Having defined the ternary relations ρ_1 and ρ_2, we can now say that a vector space *is* a mathematical structure; we denote it by

$$(\mathscr{V}; \rho_1, \rho_2)$$

Notation 2 ★★

In fact, we have glossed over the addition and multiplication of scalars, as we did originally in *Unit 22*, but these operations can be taken care of similarly.

36.2.3 Summary

36.2.3 Summary ★★

The main result of this section is the definition of a mathematical structure:

> A *mathematical structure* is a set, together with any internal n-ary relations defined on it.

We considered some particular cases of mathematical structures in the examples and exercises.

In the next section we shall examine how we can link one structure with another, and we focus our attention on those mappings which preserve structure — the morphisms.

36.3 MORPHISMS

36.3

36.3.0 Introduction

36.3.0

Introduction

* *

We hope that the reason for looking at morphisms so early in the course has become apparent to you as you worked through later units. Its existence in relation to two mathematical structures enables us to infer that some properties true in one structure are also true in the other. We made particular use of this when the structure of one of the sets was easier to handle than the structure of the other. In many units, we were quick to test for the existence of morphisms.

In this section, we shall look at the way in which we can define a morphism in terms of a general mathematical structure. But rather than jump in at the deep end, we start by looking at the types of mathematical structure with which you are familiar.

36.3.1 Structures with a Relation

36.3.1

Main Text

* *

We shall start with the relation, i.e. an internal binary relation, and, since we are interested in the way in which the morphism links two mathematical structures, we restrict our attention to the internal binary relation

$$\rho \subseteq S \times S.$$

Suppose now that we have a function

$$f: S \longmapsto T,$$

where $T = f(S)$, and that we are given an internal binary relation

$$\gamma \subseteq T \times T$$

in T. So far we have no explicit definition of a morphism involving binary relations, so we can choose what we want to do. To fox* our ideas, we consider an example.

Example 1

Example 1

Let S and T be R (or subsets of R), so that f is some real function. Let ρ and γ both represent $\leqslant$. Then we might regard it as appropriate to our general idea of morphism if $f(x) \leqslant f(y)$ when $x \leqslant y$. Consider

(i) $f: x \longmapsto x^3$ $\quad (x \in R)$

$$x \leqslant y \Rightarrow x^3 \leqslant y^3$$

(ii) $f: x \longmapsto \dfrac{1}{x}$ $\quad (x \in R, x \neq 0)$

$$x \leqslant y \not\Rightarrow \frac{1}{x} \leqslant \frac{1}{y}$$

(iii) $f: x \longmapsto \sin x$ $\quad (x \in R)$

$$x \leqslant y \not\Rightarrow \sin x \leqslant \sin y$$

(iv) $f: x \longmapsto \sin x$ $\quad (x \in [0, \pi/2])$

$$x \leqslant y \Rightarrow \sin x \leqslant \sin y$$

(v) $f: x \longmapsto \ln x$ $\quad (x \in R^+)$

$$x \leqslant y \Rightarrow \ln x \leqslant \ln y.$$

We see that (i), (iv), and (v) link the relations in domain and codomain satisfactorily, whereas (ii) and (iii) do not. ■

* Our typists are not mathematically-minded!

So we define f to be a morphism as follows.

Main Text ★★

f is a morphism of the mathematical structure $(S; \rho)$ to the mathematical structure $(f(S); \gamma)$ if whenever

Definition 1 ★★★

$$(x, y) \in \rho,$$

then

$$(f(x), f(y)) \in \gamma.$$

Example 1

Example 1

$$f : x \longmapsto \frac{1}{x} \qquad (x \in R^+)$$

is a morphism of $(R^+; \leqslant)$ to $(R^+; \geqslant)$ ■

Exercise 1

Exercise 1
(5 minutes)

When we were considering morphisms of binary operations in *Unit 3*, we considered the following problem.

Given

$$S, \circ, f,$$

can we define $\Box$ on $f(S)$ such that f is a morphism of $(S, \circ)$ to $(f(S), \Box)$?

This led us into a discussion on compatibility.

We ask the same question here. Given

$$S, \rho, f,$$

where ρ is an internal binary relation on S, can we define an internal binary relation γ on $f(S)$ such that f is a morphism of $(S; \rho)$ to $(f(S); \gamma)$?

(HINT: The answer is *not* complicated.) ■

Solution 1

Solution 1

Yes. Remember that an internal binary relation is no more than a subset of a Cartesian product of a set with itself.

Thus

$$(f(x), f(y)) \in f(S) \times f(S),$$

so we can define the binary relation γ on $f(S)$ by the rule

$$(f(x), f(y)) \in \gamma$$

whenever

$$(x, y) \in \rho.$$

The reason for the compatibility condition when we discussed binary operations arises from the fact that a binary operation is a little more than just a ternary relation — it is a ternary relation plus a few restrictions.

If we were to impose conditions on the binary relation ρ above, such as conditions which make ρ an equivalence relation, then we have to be more careful. We should have to ensure that f left the conditions unaltered. ■

36.3.2 Structures with a Binary Operation

36.3.2

Main Text
★ ★

In *Unit 3, Operations and Morphisms,* we discussed the concept of a morphism (in the context of binary operations) in some detail. We defined a morphism as follows:

A morphism is a function

$$f:(A, \circ) \longmapsto (f(A), \square)$$

such that

$$f(a_1) \square f(a_2) = f(a_1 \circ a_2) \quad \text{for all} \quad a_1, a_2 \in A.$$

By the notation $(A, \circ)$ we mean the set A, together with the (closed) binary operation $\circ$ defined on A. This was the first example of a mathematical structure (as defined in this unit) that we met in the course. In our investigation of mathematical structure in this unit, however, we have been led to think of a binary operation as a special kind of ternary relation. So rather than say that the function f maps $(A, \circ)$ to $(f(A), \square)$, we shall write the statement in the form

$$f:(A; \rho) \longmapsto (f(A); \gamma),$$

where the relations ρ and γ represent the binary operations $\circ$ and $\square$ respectively. How do we express the condition

$$f(a_1) \square f(a_2) = f(a_1 \circ a_2)$$

in terms of the ternary relations ρ and γ? We know that, if

$$(a_1, a_2, a_3) \in \rho,$$

then

$$a_1 \circ a_2 = a_3.$$

Now when an ordered triple, (x, y, z) say, belongs to γ,

$$x \square y = z.$$

So

$$(f(a_1), f(a_2), f(a_1 \circ a_2)) \in \gamma$$

whenever

$$(a_1, a_2, a_1 \circ a_2) \in \rho,$$

corresponds precisely to

$$f(a_1) \square f(a_2) = f(a_1 \circ a_2).$$

Thus we can express the condition for a function to be a morphism of one structure with a binary operation to another in the following way.

Definition 1
* * *

f is a morphism of the mathematical structure $(A\,;\rho)$ to the mathematical structure $(f(A);\gamma)$ if, whenever

$$(a_1, a_2, a_3) \in \rho,$$

then

$$(f(a_1) f(a_2), f(a_3)) \in \gamma.$$

If you look back to the previous section, you will see that this definition has striking similarities with the definition given there. We shall use these common points as a guide to a more general definition of morphisms of mathematical structures.

Exercise 1

Exercise 1
(5 minutes)

In *Unit 3, Operations and Morphisms,* we developed a test to find out whether or not a given function could be a morphism. We called it the test for *compatibility*. Express the compatibility condition in terms of a ternary relation. ■

Example 1

Example 1

Consider the set of real, differentiable functions, $\mathscr{F}$. ($\mathscr{F}$ is the set of real functions f, for which the function Df exists.) We have a *set*; let's make it into a *mathematical structure* by adding a binary operation. We shall do this in two different ways.

(i) We use addition of functions, defined by the rule

$$(f+g)(x) = f(x) + g(x) \qquad (x \in R).$$

We now have a mathematical structure $(\mathscr{F}\,;\rho)$, where ρ is the internal ternary relation corresponding to $+$. We know that the differentiation operator D is a morphism with respect to the operation $+$. This means that we can find an internal ternary relation γ, corresponding to an operation $\square$ on $D(\mathscr{F})$, such that the mathematical structure $(D(\mathscr{F});\gamma)$ has similar properties to the mathematical structure $(\mathscr{F}\,;\rho)$. The operation $\square$ is "addition of functions". Thus

$$D(f+g) = Df + Dg.$$

Expressed in terms of ternary relations, we have

$$D:(\mathscr{F}\,;\rho) \longrightarrow (D(\mathscr{F});\gamma).$$

Thus if

$$(f, g, h) \in \rho, \qquad \text{i.e.} \quad h = f + g,$$

then

$$(Df, Dg, Dh) \in \gamma, \qquad \text{i.e. } Dh = Df + Dg.$$

(ii) We use composition of functions, defined by the rule

$$f \circ g(x) = f(g(x)).$$

We now have a mathematical structure $(\mathscr{F}\,;\rho_1)$, where ρ_1 is the internal ternary relation corresponding to $\circ$. This time let's use the compatibility condition to test whether or not the differentiation operator D can be a morphism of $\mathscr{F}$ with respect to $\circ$.

(*continued on page 22*)

Solution 1

Solution 1

The compatibility condition is

if

$$f(a_1) = f(a_2)$$

and

$$f(a_3) = f(a_4),$$

then

$$f(a_1 \circ a_3) = f(a_2 \circ a_4).$$

We can express this condition in terms of the ternary relation corresponding to $\circ$ as follows.

If (a_1, a_3, a_5) and $(a_2, a_4, a_6) \in \rho$, and

$$f(a_1) = f(a_2)$$

$$f(a_3) = f(a_4),$$

then

$$f(a_5) = f(a_6).$$

■

(continued from page 21)

We shall use the compatibility condition in the form given in Solution 1.

Consider the triples

$$(x \longmapsto x^2, \quad x \longmapsto x, \quad x \longmapsto x^2),$$

$$(x \longmapsto x^2 + 1, x \longmapsto x + 2, x \longmapsto (x+2)^2 + 1),$$

which both belong to the ternary relation ρ_1 corresponding to $\circ$.

We see that

$$D(x \longmapsto x^2) = x \longmapsto 2x = D(x \longmapsto x^2 + 1)$$

and

$$D(x \longmapsto x) = x \longmapsto 1 = D(x \longmapsto x + 2).$$

But

$$D(x \longmapsto x^2) = x \longmapsto 2x \neq x \longmapsto 2x + 4 = D(x \longmapsto (x+2)^2 + 1).$$

This means that the compatibility condition for D and "composition of functions" is not satisfied. So there is no binary operation $\square$, such that

$$Df \,\square\, Dg = D(f \circ g).$$

(The formula for finding $D(f \circ g)$ that we derived in *Unit 12, Differentiation I*, section 12.2.5, is called the *chain rule*; it is:

$$D(f \circ g) = (Df) \circ g \times Dg.$$

Notice that the right-hand side of this formula contains Df and Dg, but it also contains g. When testing for compatibility, we were trying, by implication, to find a formula which contained only Df and Dg, so that the operation $\square$ could be defined as some combination of just these two.) ■

36.3.3 Structures with a Relation and a Binary Operation

36.3.3

Digression
*

This section is a slight side-track from our present investigation of morphisms of mathematical structures. Therefore you can omit this section without losing the threads of the discussion.

Let us take up one of the important points in *Unit 19, Relations.* Suppose we have a mathematical structure $(A; \rho_1, \rho)$ where

ρ_1 is the relation corresponding to a closed binary operation $\circ$ on A,

and

ρ is an equivalence relation on A.

Let us suppose that the binary operation $\circ$ is compatible with the equivalence relation ρ.* That is, whenever

$$(x_1, x_2) \in \rho \quad \text{and} \quad (y_1, y_2) \in \rho$$

then

$$(x_1 \circ y_1, x_2 \circ y_2) \in \rho.$$

From a set with an equivalence relation defined on it, we were led to the concept of the quotient set A/ρ, the set of equivalence classes of ρ. The fact that $\circ$ and ρ are compatible is particularly useful in this case, for we can start to build up a mathematical structure with the set A/ρ. We define the binary operation $\square$ on the set of equivalence classes by the rule

$$[x] \square [y] = [x \circ y],$$

where $[x]$ denotes the equivalence class to which x belongs. The compatibility of $\circ$ and ρ ensures that this definition is sound.

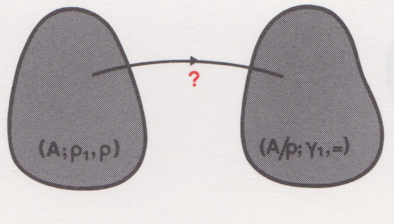
We now have the mathematical structure $(A/\rho; \gamma_1, \gamma)$, where γ_1 is the relation corresponding to $\square$ on A/ρ. But what is γ? We find that γ is the equivalence relation "equality of sets". For, whenever $(x, y) \in \rho$ in A, then

$$[x] = [y] \text{ in } A/\rho,$$

i.e.

$$([x], [y]) \in \text{"="} \text{ in } A/\rho.$$

Thus starting from $(A; \rho_1, \rho)$, we have produced another mathematical structure $(A/\rho; \gamma_1, =)$ which is structurally similar, but very much simpler.



As we have said "structurally similar", we should state the morphism which connects these two structures. It is, of course, the natural mapping n, defined by

$$n: x \longmapsto [x] \qquad (x \in A).$$

* We have written the condition in terms of our present notation, and not that of *Unit 19.*

Exercise 1

Exercise 1
(3 minutes)

Given that $\circ$ and ρ are compatible, check that the natural mapping *is* a morphism of the mathematical structure $(A; \rho_1, \rho)$ to the mathematical structure $(A/\rho; \gamma_1, =)$. You need to verify that:

(i) $$n(x) \square n(y) = n(x \circ y),$$

which is the condition for $n:(A, \circ) \longmapsto (A/\rho, \square)$ to be a morphism;

(ii) whenever $(x, y) \in \rho$, then $(n(x), n(y)) \in =$, which is the condition for $n:(A; \rho) \longmapsto (A/\rho; =)$ to be a morphism. ■

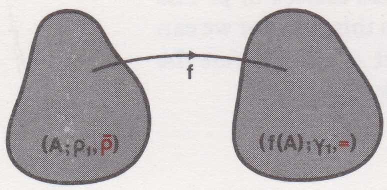
Discussion
★★

We have just seen how, given the mathematical structure $(A; \rho_1, \rho)$, where ρ is an equivalence relation, and ρ_1 is the relation corresponding to the operation $\circ$ where $\circ$ and ρ are compatible, then we have a morphism, which is the natural mapping, n.

The situation may present itself the other way round. We may start with the morphism

$$f:(A; \rho_1) \longmapsto (f(A); \gamma_1).$$

We can now create the equivalence relation — we call it the natural equivalence relation, $\bar{\rho}$.



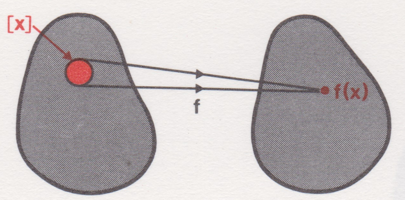
We do this in such a way that the morphism

$$f:(A, \rho_1) \longmapsto (f(A), \gamma_1)$$

is a morphism for the extended structures $(A; \rho_1, \bar{\rho})$ and $(f(A); \gamma_1, =)$. We define $\bar{\rho}$ as follows

$$(x, y) \in \bar{\rho} \quad \text{whenever} \quad f(x) = f(y).$$

That is, the equivalence class $[x]$ of x is the set of elements of A which map to the single element $f(x)$.



Main Text
★★

We suggest that you re-read Example 19.2.3.3 of *Unit 19*.
When we have a morphism of one mathematical structure to another

$$f:(A; \rho_1) \longmapsto (f(A); \gamma_1),$$

the method we have given for defining the natural equivalence relation proves particularly useful if the mathematical structure $(A; \rho_1)$ is of a particular type. For example, if $(A, \circ)$ is a *group*. In this case, the fact that f is a morphism means that we know that $(f(A), \square)$ is also a group.

In *Unit 33, Groups II* we defined

$$\text{the kernel of } f = \{x \in A : f(x) = e_{f(A)}\}$$

in terms of the natural equivalence relation $\bar{\rho}$.

The kernel of f is $[e_A]$, the equivalence class of the identity element of A (because we know that $e_A \longmapsto e_{f(A)}$). In *Unit 33* we saw that the kernel of f is a normal subgroup of $(A, \circ)$, and that any normal subgroup of $(A, \circ)$ is the kernel of some morphism f.

Example 1

Example 1

In *Unit 33, Groups II*, section 33.1.4, we proved that if $(H, \circ)$ is a subgroup of $(G, \circ)$, then any two cosets of H in G are either identical or have no common elements.

We can obtain this result in another way by using an equivalence relation.

We have a group $(G, \circ)$ with a subgroup $(H, \circ)$. We now define a binary relation ρ on G. We shall say that

$$x\rho y \quad \text{if and only if} \quad x^{-1} \circ y \in H.$$

We have the following results for any $x, y, z \in G$.

(i) $x^{-1} \circ x = e_g = e_h \in H$, so $x\rho x$.

(ii) If $x\rho y$, then $x^{-1} \circ y \in H$. Since $(H, \circ)$ is a group, $x^{-1} \circ y$ has a *unique* inverse in H; but

$$(x^{-1} \circ y) \circ (y^{-1} \circ x) = e_h$$

so

$$(x^{-1} \circ y)^{-1} = y^{-1} \circ x \in H$$

i.e.

$$x\rho y \Rightarrow y\rho x.$$

(iii) If $x\rho y$ and $y\rho z$, then

$$x^{-1} \circ y \quad \text{and} \quad y^{-1} \circ z \in H.$$

Since $(H, \circ)$ is a group,

$$(x^{-1} \circ y) \circ (y^{-1} \circ z) \in H$$

i.e.

$$x^{-1} \circ (y \circ y^{-1}) \circ z \in H$$

i.e.

$$x^{-1} \circ e_g \circ z \in H$$

i.e.

$$x^{-1} \circ z \in H$$

$$x\rho y \text{ and } y\rho z \Rightarrow x\rho z.$$

We have shown that ρ is an equivalence relation on G, so it partitions G into disjoint equivalence classes, called left cosets of G mod H. We see that x and y belong to the same left coset if and only if

$$x^{-1} \circ y \in H$$

i.e.

$$y \in xH.$$

In *Unit 33* we also showed that if $(H, \circ)$ is a normal subgroup of $(G, \circ)$, then the set of cosets of H in G forms a group $(G/H, \Box)$. The natural mapping

$$n : (G; \rho_1, \rho) \longmapsto (G/H; \gamma_1, =)$$

is a morphism, where ρ_1 is the relation corresponding to $\circ$ on G, and γ_1 is the relation corresponding to $\Box$ on G/H. ■

Solution 1

Solution 1

(i) $n(x) \square n(y) = [x] \square [y]$

$$= [x \circ y]$$

$$= n(x \circ y).$$

(ii) $n(x) = [x]$ and $n(y) = [y]$. But, since $(x, y) \in \rho$, x and y belong to the same equivalence class, so we know that

$$[x] = [y].$$ ■

36.3.4 Morphisms of a Vector Space

36.3.4

Discussion
* *

In section 36.2.2 we saw that, in order to express a vector space as a mathematical structure, we had to combine the set of vectors V with the set of scalars R. This gave rise to the set $\mathscr{V} = V \cup R$. The two important operations connected with a vector space are addition of vectors, and multiplication of a vector by a scalar. We expressed these operations in terms of the ternary relations ρ_1 and ρ_2 respectively on $\mathscr{V}$.

Let us now examine the way in which a morphism relates one vector space to another. Suppose that f is a function

$$f: V \longmapsto U,$$

where V and U are sets of vectors. We add the necessary structure to V and U in order to make them vector spaces.

Thus

$$(\mathscr{V}; \rho_1, \rho_2)$$

is the vector space with the set of vectors V, and

$$(\mathscr{U}, \gamma_1, \gamma_2)$$

is a vector space with the set of vectors U. Also,

γ_1 is the ternary relation expressing addition of vectors in $\mathscr{U}$,

γ_2 is the ternary relation expressing scalar multiplication in $\mathscr{U}$.

We shall look first at the way in which f affects the ternary relation ρ_1. We therefore look at a typical ordered triple of vectors $(\underset{\sim}{v}_1, \underset{\sim}{v}_2, \underset{\sim}{v}_3)$ which belongs to ρ_1. The corresponding triple in γ_1 is $(f(\underset{\sim}{v}_1), f(\underset{\sim}{v}_2), f(\underset{\sim}{v}_3))$. Now if $(f(\underset{\sim}{v}_1), f(\underset{\sim}{v}_2), f(\underset{\sim}{v}_3))$ belongs to the ternary relation γ_1, then

$$f(\underset{\sim}{v}_1) + f(\underset{\sim}{v}_2) = f(\underset{\sim}{v}_3),$$

that is,

$$f(\underset{\sim}{v}_1) + f(\underset{\sim}{v}_2) = f(\underset{\sim}{v}_1 + \underset{\sim}{v}_2).$$

Thus f will be a morphism of the mathematical structure $(\mathscr{V}; \rho_1)$ to $(\mathscr{U}; \gamma_1)$ if this last equation holds. This is the first of the two conditions we impose on a vector space morphism.

Let us now turn to the second ternary relation ρ_2. Here, the ordered triple which is a typical element of ρ_2 is

$$(\lambda, \underset{\sim}{v}_1, \underset{\sim}{v}_2).$$

What is the "image" of this triple under f? Not all the elements of the triple belong to the domain of f, for we defined f to be the function

$$f: V \longmapsto U.$$

We take the triple corresponding to $(\lambda, \underset{\sim}{v}_1, \underset{\sim}{v}_2)$ in ρ_2 to be $(\lambda, f(\underset{\sim}{v}_1), f(\underset{\sim}{v}_2))$ in γ_2. If this triple is to be an element of the ternary relation γ_2 on the image vector space, then we require that

$$\lambda f(\underset{\sim}{v}_1) = f(\underset{\sim}{v}_2),$$

and since

$$\lambda \underset{\sim}{v}_1 = \underset{\sim}{v}_2,$$

this becomes

$$\lambda f(\underset{\sim}{v}_1) = f(\lambda \underset{\sim}{v}_1)$$

This is just the second condition that we impose on a vector space morphism.

Let us sum up these two results in the way that we did for morphisms of structures with binary relations and binary operations.

Let $(\mathscr{V}; \rho_1, \rho_2)$ and $(\mathscr{U}; \gamma_1, \gamma_2)$ be two mathematical structures and ρ_1, ρ_2 and γ_1, γ_2 be internal ternary relations expressing addition of vectors and multiplication of a vector by a scalar respectively. Then f is a morphism of the mathematical structure $(\mathscr{V}; \rho_1, \rho_2)$ to the mathematical structure $(\mathscr{U}; \gamma_1, \gamma_2)$ if whenever

$$(\underset{\sim}{v}_1, \underset{\sim}{v}_2, \underset{\sim}{v}_3) \in \rho_1$$

and

$$(\lambda, \underset{\sim}{v}_1, \underset{\sim}{v}_2) \in \rho_2,$$

then

$$(f(\underset{\sim}{v}_1), f(\underset{\sim}{v}_2), f(\underset{\sim}{v}_3)) \in \gamma_1$$

and

$$(\lambda, f(\underset{\sim}{v}_1), f(\underset{\sim}{v}_2)) \in \gamma_2.$$

We conclude this section by looking at a particular example of a mathematical structure which is a vector space.

Example 1 **Example 1**

In Unit 7, Sequences and Limits I, we discussed . . . (guess what!) It is easy to show that the set of all (real) infinite sequences forms a vector space, but we shall restrict our attention to those infinite sequences which are convergent.

Let $U = \{\underset{\sim}{u} : \underset{\sim}{u} \text{ is a convergent sequence}\}$. We shall think of the sequence $\underset{\sim}{u}$ as a vector. In *Unit 7*, we defined addition of sequences as follows.

$$\underset{\sim}{u} + \underset{\sim}{v}$$

is the sequence

$$u_1 + v_1, u_2 + v_2, \ldots, u_k + v_k, \ldots$$

This defines addition of vectors for us.

We can also introduce a form of multiplication by a scalar. If $\lambda \in R$, then we define

$$\lambda \underset{\sim}{u}$$

to be the sequence

$$\lambda u_1, \lambda u_2, \ldots, \lambda u_k, \ldots$$

There is little point in wading through all ten axioms of a vector space in order to verify that they are all satisfied. Take our word for it — we

have not deceived you (very) often! But note that the identity element for the group of vectors is the zero sequence $\underset{\sim}{0}$:

$$0, 0, 0, \ldots, 0, \ldots$$

We now have a vector space, which is a mathematical structure

$$(\mathscr{U}; \rho_1, \rho_2),$$

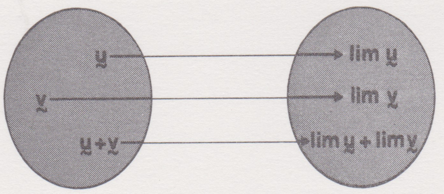
where the relation ρ_1 corresponds to addition of vectors (i.e. sequences) and ρ_2 corresponds to multiplication of a vector by a scalar.

Since we restricted U to be the set of convergent sequences, we are not surprised to find that a morphism lurks in the concept of a limit. We can think of the limit process as defining a function

$$\lim : U \longrightarrow R,$$

where lim maps each convergent sequence to the real number which is its limit.

(Note that since U consists only of convergent sequences, $\lim \underset{\sim}{u}$ exists for each $\underset{\sim}{u} \in U$, *and* $\lim \underset{\sim}{u}$ is unique. Thus $\lim : U \longrightarrow R$ *is* a function.) In *Unit 7* we showed that lim is a *morphism with respect to the addition of vectors*:



$$\lim(\underset{\sim}{u} + \underset{\sim}{v}) = \lim \underset{\sim}{u} + \lim \underset{\sim}{v}.$$

It is also easy to show that

$$\lambda \lim \underset{\sim}{u} = \lim(\lambda \underset{\sim}{u}) \qquad (\lambda \in R).$$

So lim is a vector space morphism (linear transformation).

We showed in *Unit 23, Linear Algebra II* that the image of a vector space under a morphism is itself a vector space. So the set of real numbers R forms a vector space "over itself" (i.e. the scalars are also real numbers).

In this vector space,

(i) the vectors are simply real numbers, and addition of vectors is "ordinary" addition;
(ii) the scalars are also real numbers, and multiplication of a vector by a scalar is "ordinary" multiplication.

So under the morphism lim:

$(U, +)$ corresponds to $(R, +)$

and

multiplication by a scalar corresponds to $\times$.

Let's sum up these results in terms of mathematical structures:

$$\lim : (\mathscr{U}; \rho_1, \rho_2) \longrightarrow (R; +, \times)$$

For the ternary relation, ρ_1, corresponding to addition of vectors, we have:

$(\underset{\sim}{u}, \underset{\sim}{v}, \underset{\sim}{w})$ corresponds to $(\lim \underset{\sim}{u}, \lim \underset{\sim}{v}, \lim \underset{\sim}{w})$,

so

$$\lim \underset{\sim}{u} + \lim \underset{\sim}{v} = \lim \underset{\sim}{w}$$
$$= \lim (\underset{\sim}{u} + \underset{\sim}{v}).$$

For the ternary relation, ρ_2, corresponding to multiplication of a vector by a scalar, we have:

$(\lambda, \underset{\sim}{u}, \underset{\sim}{v})$ corresponds to $(\lambda, \lim \underset{\sim}{u}, \lim \underset{\sim}{v})$,

so

$$\lambda \lim \underset{\sim}{u} = \lim \underset{\sim}{v}$$
$$= \lim (\lambda \underset{\sim}{u}).$$

Finally, since we have shown that lim is a morphism having a vector space as its domain, the kernel of the morphism is also a vector space. How can we find it? Let's follow the argument in section 36.3.3. We start by defining the natural equivalence relation $\bar{\rho}$ in $(\mathscr{U}; \rho_1, \rho_2)$ by the rule

$(\underset{\sim}{u}, \underset{\sim}{v}) \in \bar{\rho}$ if $\lim \underset{\sim}{u} = \lim \underset{\sim}{v}$.

Thus the equivalence class $[\underset{\sim}{u}]$ contains all those sequences which have the same limit as $\underset{\sim}{u}$. The special equivalence class that is the kernel of lim is now seen to be $[\underset{\sim}{0}]$ i.e., the set of all sequences which have limit 0.

Although it is not obvious from what we have said, the equivalence relation $\bar{\rho}$ has important ramifications in the study of analysis — we shall discuss this point in the next section. ■

36.3.5 The Real Numbers

36.3.5

Digression
*

As with section 36.3.3, this section is also a slight side-track from the main theme of morphisms of mathematical structures. You can therefore omit this section without fear of missing anything which is central to the argument. If however you decide to read it, you should make sure that you have understood both section 36.3.3 and Example 36.3.4.1. The reason for including this section in the text is to show how we can use many different ideas about mathematical structure to help us in a particular case.

In *Unit 34*, *Number Systems* we developed a method of building up the sets Z, Q and R from the simplest set of numbers, N, the set of natural numbers. The process that we used there was the following.

$$N \xrightarrow{\text{Step 1}} Z \xrightarrow{\text{Step 2}} Q \xrightarrow{\text{Step 3}} R$$

We made Step 1 by considering the mathematical structure $(N \times N; \rho, \rho', \rho_1)$, where ρ is the relation corresponding to addition of number pairs, ρ' is the relation corresponding to "multiplication" of number pairs, and ρ_1 is the equivalence relation:

$$(m, n)\rho_1(m', n')$$

if and only if

$$m + n' = m' + n.$$

(See *Unit 34*, sections 34.2.1, 2.)

We used a morphism to "identify" $N \times N$ with the set of integers Z. We found that

(i) $(Z, +)$ is an Abelian group;
(ii) $(Z, \times)$ is a semi-group;
(iii) $\times$ is distributive over $+$.

A mathematical structure with analogous properties is called a ring.

We made Step 2 by considering the mathematical structure $(Z \times Z; \rho, \rho', \rho_2)$, where ρ is the relation corresponding to addition of integer pairs, ρ' is the relation corresponding to "multiplication" of integer pairs, and ρ_2 is the equivalence relation:

$$(m, n)\rho_2(m', n')$$

if and only if

$$mn' = m'n.$$

(See *Unit 34*, sections 34.3.1, 2.)

We used a morphism to "identify" $Z \times Z$ with the set of rationals Q. We found that

(i) $(Q, +)$ is an Abelian group;
(ii) $(Q_1, \times)$ is an Abelian group, where Q_1 is the set of non-zero rationals;
(iii) $\times$ is distributive over $+$.

A mathematical structure with analogous properties is called a field.

To make the final step from Q to R, we used the concept of upper and lower bounds of sets of rationals. But this is not the only method available for making Step 3. We can also use the ideas discussed in Example 36.3.4.1.

In *Unit 7*, *Sequences and Limits I*, we showed how we could think of a sequence $\underset{\sim}{u}$ in terms of a function

$$f: k \longmapsto u_k \qquad (k \in N).$$

We know all about the domain of f, but what about its codomain? Since we have set up Q, the set of rational numbers, we may suppose that f has Q as its codomain. That is, we shall restrict our attention to sequences whose elements are rational numbers. In fact, let's go one step further and restrict the sequences to those which satisfy the particular property that the difference

$$|u_m - u_n| \tag{1}$$

becomes very small when both m and n become very large. In plain terms, we can state this property by saying that the difference between any two elements which occur "well on" in the sequence gets "very small". Such a sequence is

$$\underset{\sim}{u} = \{3, 3.1, 3.14, 3.141, 3.1415, 3.14159, \ldots\},$$

whose elements are successive finite decimal—and hence rational—approximations to π. For this sequence, any two elements after the 10th will differ by less than $1/10^9$.

If we denote this set of sequences by U, and take the rationals, Q, to be the field of scalars, then the set

$$\mathscr{U} = U \cup Q$$

together with suitable n-ary relations ρ_1 and ρ_2 i.e. $(U; \rho_1, \rho_2)$ forms a mathematical structure which is a vector space.

Can we say that the sequences satisfying property (1) are convergent? If the answer is "yes", then we can apply the morphism lim to the set U:

$$\lim : U \longrightarrow ?$$

But what is the codomain of lim? It's not Q, because from the example $\underset{\sim}{u}$, given above, we know

$$\lim \underset{\sim}{u} \quad \text{to be } \pi$$

and π is not a rational number. In fact the codomain of lim turns out to be the set R which we are trying to define. But since we "don't yet know"

about R, we cannot define lim. So we cannot call the sequence $\underset{\sim}{u}$ convergent unless $\lim \underset{\sim}{u} \in Q$.

What we do is use the process discussed in section 36.3.3, and arrange things so that lim becomes the natural mapping associated with a particular equivalence relation. We define a binary relation by the rule

$$(\underset{\sim}{u}, \underset{\sim}{v}) \in \bar{\rho}$$

whenever the sequence $\underset{\sim}{u} - \underset{\sim}{v}$ is convergent with limit 0. Note that since 0 is a rational number, this definition *can* be made. Without going into details, we shall assume that the binary relation so defined, is, in fact, an equivalence relation. We now have the mathematical structure

$$(\mathscr{U}; \rho_1, \rho_2, \bar{\rho}),$$

and can set up another one from it by considering the quotient set $\mathscr{U}/\rho$. The new structure is

$$(\mathscr{U}/\rho; \gamma_1, \gamma_2, =)$$

when γ_1 and γ_2 correspond to ρ_1 and ρ_2 in the original structure. Let's look a bit more closely at the elements of $\mathscr{U}/\rho$. They are equivalence classes of sequences, denoted by

$$[\underset{\sim}{u}],$$

and each member of this particular equivalence class will be a sequence, $\underset{\sim}{v}$ say, such that

$$\lim (\underset{\sim}{u} - \underset{\sim}{v}) = 0.$$

The final step of this method of constructing the set R is an easy one. We simply say that

$$R = \mathscr{U}/\rho,$$

that is, we define a real number to be an equivalence class of rational sequences. The reason for making this step is made clearer if we look back to our problem of not being able to define the morphism lim since we didn't know its codomain. Owing to the way in which we defined the equivalence relation $\bar{\rho}$, we can make the following statements about the natural mapping associated with $\bar{\rho}$:

(i) $n : \underset{\sim}{u} \longmapsto [\underset{\sim}{u}]$,
and since $R = \mathscr{U}/\rho$, $[\underset{\sim}{u}] \in R$, we can write
(ii) $n : \underset{\sim}{u} \longmapsto a$, some real number a.

Finally, since we want to give some meaning to the real number a, we find that this is best done if we call a the *limit of the sequence* $\underset{\sim}{u}$.

The reason why this is a sensible thing to do is that

if

$$\underset{\sim}{u} \quad \text{and} \quad \underset{\sim}{v} \in [\underset{\sim}{u}],$$

then

$$\lim (\underset{\sim}{u} - \underset{\sim}{v}) = 0.$$

So that

$$\lim \underset{\sim}{u} - \lim \underset{\sim}{v} = 0$$

i.e.

$$\lim \underset{\sim}{u} = \lim \underset{\sim}{v}.$$

Thus a is the same number whichever sequence in the equivalence class $[\underset{\sim}{u}]$ is chosen.

All we have said here can be summed up as follows.

The set of real numbers can be defined as the set of limit points of all sequences of rational numbers which satisfy property (1).

36.3.6 Morphisms of a Mathematical Structure

36.3.6

Main Text

Finally, let us see how the concept of a morphism fits in with the general concept of a mathematical structure. We do this by looking at what happens when a function is applied to an internal n-ary relation. In looking for the definition of a morphism we follow the same process that we used for morphisms of structures with a relation or a binary operation. We gave the following summary for the particular case where ρ and ρ' correspond to binary operations. (See page 21.)

> f is a morphism of the mathematical structure $(S; \rho)$ to the mathematical structure $(T; \rho')$ if, whenever $(x, y, z) \in \rho$, then
>
> $(f(x), f(y), f(z)) \in \rho'$.

A slight problem presented itself when we came to look at the morphism of a vector space, and particularly its effect upon multiplication of a vector by a scalar (see page 26), because the scalar λ does not belong to the domain of f.

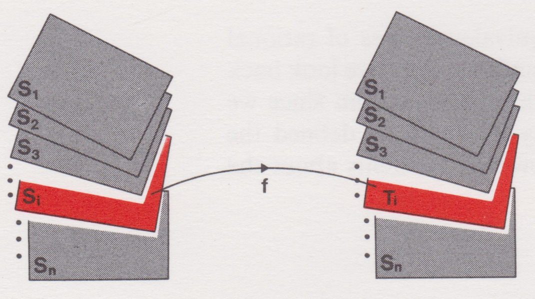
We overcame this problem by taking the triple corresponding to $(\lambda, \underset{\sim}{v}_1, \underset{\sim}{v}_2)$ in ρ to be $(\lambda, f(\underset{\sim}{v}_1), f(\underset{\sim}{v}_2))$ in ρ'.

This method can be extended to the general mathematical structure. We'll start with our set S, which we now consider to be a union of sets $S_1, S_2, \ldots, S_n$. Now suppose that f is a function

$$f: S_i \longrightarrow T_i, \quad \text{where } i \text{ is a particular integer} \quad 1 \leqslant i \leqslant n.$$

We now define the other set T to be

$$T = S_1 \cup \cdots \cup S_{i-1} \cup T_i \cup S_{i+1} \cup \cdots \cup S_n.$$



The next step is to "extend" f to $\bar{f}$ so that the domain of $\bar{f}$ is S. We do this by letting $\bar{f}$ be the identity function outside S_i. We define the function $\bar{f}$ as follows:

$$\bar{f}: s_i \longmapsto f(s_i) \qquad (s_i \in S_i),$$

$$\bar{f}: s_j \longmapsto s_j \qquad (s_j \in S, s_j \notin S_i).$$

We can, by analogy, define the morphism for the case of the general mathematical structure.

Definition 1

The function $f: S_i \longrightarrow T_i$ is a morphism of the mathematical structure $(S; \rho_1, \ldots, \rho_m)$ to the mathematical structure $(T; \gamma_1, \ldots, \gamma_m)$ if, for every internal n-ary relation ρ_i defined on S, whenever

$$(x_1, x_2, \ldots, x_n) \in \rho_i,$$

then

$$(\bar{f}(x_1), \bar{f}(x_2), \ldots, \bar{f}(x_n)) \in \gamma_i,$$

where $\bar{f}$ is the "extension" of f, defined above.

36.4 An End or a Beginning?

36.4

End?
*

One aim of this unit was to show how the concept of mathematical structure gathers together many of the topics that we have covered in the Foundation Course. If the course has to end somewhere, where better than in a discussion of a *concept* which unites so much of the content of the course? But we have not followed our investigation of mathematical structures with the sole purpose of ending the course. In fact the approach has shown that all we have done is to set the stage for a new and more rigorous development of mathematics. We have made a new beginning to what must now be left to another time . . . .

New Beginning
* * *

"This is not the end. It is not even the beginning of the end. But it is, perhaps, the end of the beginning."

Sir Winston Churchill
Speech at the Mansion House
10 November 1942
(Of the Battle of Egypt)

Unit No.		Title of Text
1		Functions
2		Errors and Accuracy
3		Operations and Morphisms
4		Finite Differences
5	NO TEXT	
6		Inequalities
7		Sequences and Limits I
8		Computing I
9		Integration I
10	NO TEXT	
11		Logic I — Boolean Algebra
12		Differentiation I
13		Integration II
14		Sequences and Limits II
15		Differentiation II
16		Probability and Statistics I
17		Logic II — Proof
18		Probability and Statistics II
19		Relations
20		Computing II
21		Probability and Statistics III
22		Linear Algebra I
23		Linear Algebra II
24		Differential Equations I
25	NO TEXT	
26		Linear Algebra III
27		Complex Numbers I
28		Linear Algebra IV
29		Complex Numbers II
30		Groups I
31		Differential Equations II
32	NO TEXT	
33		Groups II
34		Number Systems
35		Topology
36		Mathematical Structures